Tiling-harmonic conjugate functions

Harish Vemuri
mentored by: Prof. Sergiy Merenkov, CCNY-CUNY

6th Annual PRIMES Conference

May 21, 2016

Basic Definitions 1

Definition

A *square tiling* of a region $R \in \mathbb{R}^2$ is a finite collection of squares with disjoint interiors, whose sides are parallel to the x and y axes, whose vertices have integer coordinates, and whose union is R. A *regular square tiling* of a region is a square tiling of the region in which all of the squares have side length 1.

Basic Definitions 1

Definition

A *square tiling* of a region $R \in \mathbb{R}^2$ is a finite collection of squares with disjoint interiors, whose sides are parallel to the x and y axes, whose vertices have integer coordinates, and whose union is R. A *regular square tiling* of a region is a square tiling of the region in which all of the squares have side length 1.

Definition

A *subsquare* of a square tiling T is one of the squares in the collection of squares that composes T.

Basic Definitions 2

Definition

Given some function f defined on the vertices of a square tiling T, the *oscillation* of a subsquare of T is the difference between the maximum and minimum values that f takes on the vertices of this subsquare.

Basic Definitions 2

Definition

Given some function f defined on the vertices of a square tiling T, the *oscillation* of a subsquare of T is the difference between the maximum and minimum values that f takes on the vertices of this subsquare.

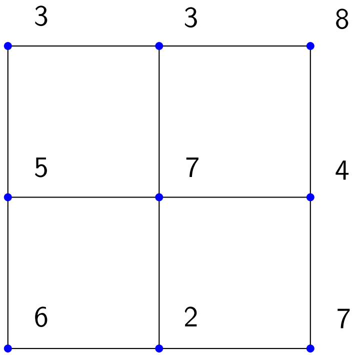
Definition

Given a function f defined on the vertices of a square tiling T, the *Energy* of the tiling $E(f, T)$ is

$$E(f, T) = \sum_{S \in T} \operatorname{osc}^2(S)$$

where the sum is taken over all subsquares S of T.

Energy Calculation Example



Energy Calculation Example

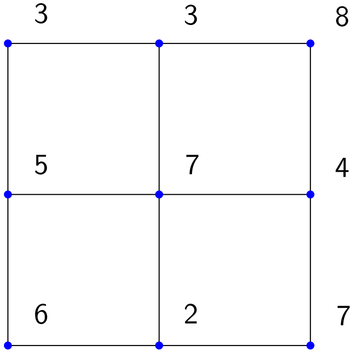


Figure : $E(f, T) = (7-3)^2 + (8-3)^2 + (7-2)^2 + (7-2)^2 = 91$

Tiling-Harmonic Definition

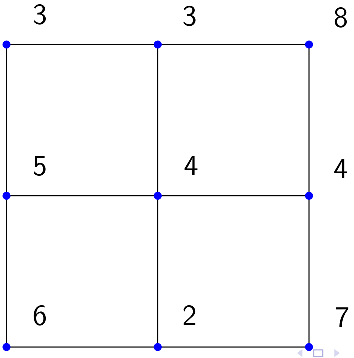
Definition (Tiling Harmonic Functions)

For a square tiling T, and the set S of all real functions defined on the vertices of T with some fixed boundary values, the *tiling harmonic function* for T and this particular set of of boundary values is the member of S with the minimum energy.

Tiling-Harmonic Definition

Definition (Tiling Harmonic Functions)

For a square tiling T, and the set S of all real functions defined on the vertices of T with some fixed boundary values, the *tiling harmonic function* for T and this particular set of of boundary values is the member of S with the minimum energy.



Graph-Harmonic Definition

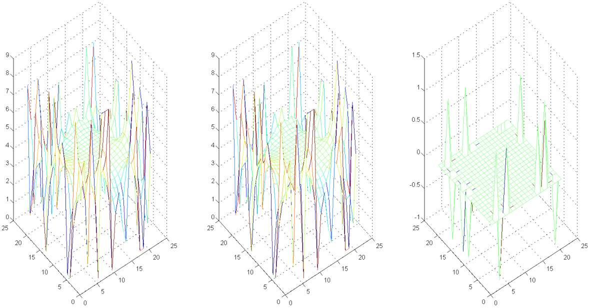
Definition (Graph Harmonic Functions)

A *graph harmonic* function f on a tiling T satisfies the property that for each interior vertex of the tiling, the value of f is the average of the values of f at the neighboring vertices.

Graph-Harmonic Definition

Definition (Graph Harmonic Functions)

A *graph harmonic* function f on a tiling T satisfies the property that for each interior vertex of the tiling, the value of f is the average of the values of f at the neighboring vertices.



Similarities between graph harmonic and tiling harmonic functions

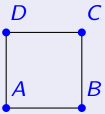
Graph-Harmonic Conjugates

Definition (Graph Analytic Function)

A graph-analytic function is one that satisfies

$$f(A) + if(B) + i^2 f(C) + i^3 f(D) = 0$$

for all subsquares of the (infinite) regular square tiling using vertex labelling as shown in the diagram.



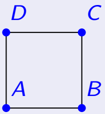
Graph-Harmonic Conjugates

Definition (Graph Analytic Function)

A graph-analytic function is one that satisfies

$$f(A) + if(B) + i^2 f(C) + i^3 f(D) = 0$$

for all subsquares of the (infinite) regular square tiling using vertex labelling as shown in the diagram.



Definition (Graph Harmonic Conjugate)

Given a graph harmonic function u, its *graph harmonic conjugate* is defined as the graph harmonic function v such that $u + iv$ is graph analytic.

Cauchy Riemann Equations

- A graph harmonic function $u(x, y)$ and its conjugate $v(x, y)$ must satisfy the discrete Cauchy Riemann equations

$$u(x+1, y) - u(x, y) = v(x, y) - v(x, y-1)$$

$$u(x, y+1) - u(x, y) = v(x-1, y) - v(x, y)$$

Cauchy Riemann Equations

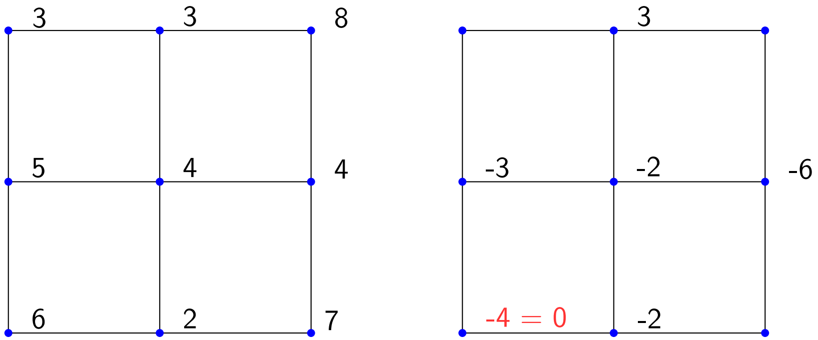
- A graph harmonic function $u(x, y)$ and its conjugate $v(x, y)$ must satisfy the discrete Cauchy Riemann equations

$$u(x+1, y) - u(x, y) = v(x, y) - v(x, y-1)$$

$$u(x, y+1) - u(x, y) = v(x-1, y) - v(x, y)$$

- The Cauchy Riemann equations from the graph harmonic case run into contradictions in the tiling harmonic case.

Cauchy Riemann Contradictions Example



A First Approach

- A possible fix for these contradictions:
  - We can conjugate the boundary values of a tiling harmonic function T and fill in the interior vertices to match the tiling harmonic definition for these boundary values.

A First Approach

- A possible fix for these contradictions:
  - We can conjugate the boundary values of a tiling harmonic function T and fill in the interior vertices to match the tiling harmonic definition for these boundary values.
- The process to conjugate a tiling harmonic function T in this manner:

A First Approach

- A possible fix for these contradictions:
  - We can conjugate the boundary values of a tiling harmonic function T and fill in the interior vertices to match the tiling harmonic definition for these boundary values.
- The process to conjugate a tiling harmonic function T in this manner:

  1. Consider the graph harmonic function G with the same boundary values as T.

A First Approach

- A possible fix for these contradictions:
  - We can conjugate the boundary values of a tiling harmonic function T and fill in the interior vertices to match the tiling harmonic definition for these boundary values.
- The process to conjugate a tiling harmonic function T in this manner:
  1. Consider the graph harmonic function G with the same boundary values as T.
  2. Use the Cauchy Riemann Equations to compute the graph harmonic conjugate function G' of G.

A First Approach

- A possible fix for these contradictions:
  - We can conjugate the boundary values of a tiling harmonic function T and fill in the interior vertices to match the tiling harmonic definition for these boundary values.
- The process to conjugate a tiling harmonic function T in this manner:

  1. Consider the graph harmonic function G with the same boundary values as T.
  2. Use the Cauchy Riemann Equations to compute the graph harmonic conjugate function G' of G.
  3. Compute the tiling harmonic function T' whose boundary values match those of G'. This will be the tiling harmonic conjugate of T.

Expanding Boundary Convergence

Conjecture

Given a sequence of sets of bounded boundary values $B_1, B_2, B_3, \ldots$ where $|B_{i+1}| > |B_i|$, and some compact set S. Then

$$\lim_{i \to \infty} T(B_i, S) - G(B_i, S) = 0$$

where $T(B_i, S)$ and $G(B_i, S)$ are the tiling and graph harmonic functions, respectively, with boundary values B_i evaluated at corresponding values in S, and the convergence is uniform on compacta.

Expanding Boundary Convergence

Conjecture

Given a sequence of sets of bounded boundary values $B_1, B_2, B_3, \ldots$ where $|B_{i+1}| > |B_i|$, and some compact set S. Then

$$\lim_{i\to\infty} T(B_i, S) - G(B_i, S) = 0$$

where $T(B_i, S)$ and $G(B_i, S)$ are the tiling and graph harmonic functions, respectively, with boundary values B_i evaluated at corresponding values in S, and the convergence is uniform on compacta.

- This conjecture is supported by large grid comparisons of tiling harmonic and graph harmonic functions with the same boundary values.

Expanding Boundary Convergence

Conjecture

Given a sequence of sets of bounded boundary values $B_1, B_2, B_3, \dots$ where $|B_{i+1}| > |B_i|$, and some compact set S. Then

$$\lim_{i \to \infty} T(B_i, S) - G(B_i, S) = 0$$

where $T(B_i, S)$ and $G(B_i, S)$ are the tiling and graph harmonic functions, respectively, with boundary values B_i evaluated at corresponding values in S, and the convergence is uniform on compacta.

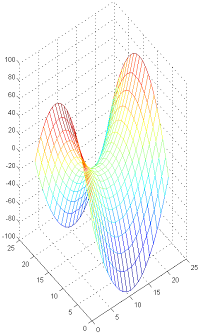
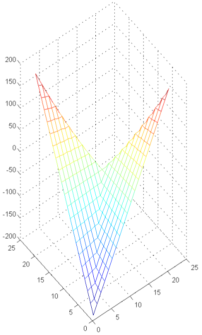
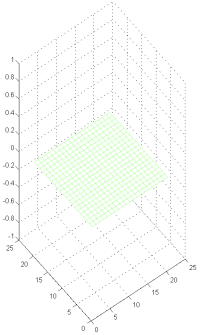
- This conjecture is supported by large grid comparisons of tiling harmonic and graph harmonic functions with the same boundary values.
- If this conjecture is true, the aforementioned conjugation process would be effective in relating tiling harmonic functions to graph analytic functions.

A Pseudo-Conjugate 1

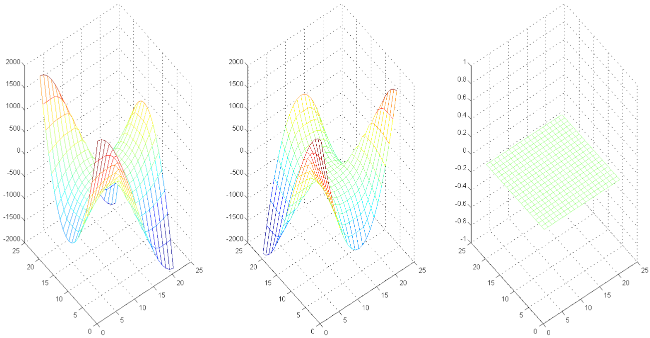
- Similar to the previous approach, it is interesting to consider the special case of tiling harmonic functions whose boundary functions are classical harmonic conjugates

A Pseudo-Conjugate 1

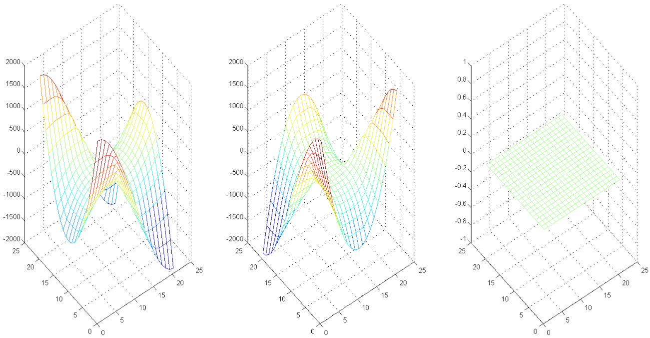
- Similar to the previous approach, it is interesting to consider the special case of tiling harmonic functions whose boundary functions are classical harmonic conjugates
- An interesting phenomenon occurs when considering the pair of tiling harmonic functions with the boundary functions $u(x, y) = x^2 - y^2, v(x, y) = 2xy$ and the pair with boundary functions $u(x, y) = x^3 - 3xy^2, v(x, y) = 3x^2y - y^3$.

A Pseudo-Conjugate 2



A Pseudo-Conjugate 2



- Rotating the second function by 90 degrees and adding to the first function gives a graph harmonic function in both cases!

Existence of Pseudo-Conjugate

Conjecture

Given a tiling harmonic function $u(x, y)$ on some regular square tiling, there exists another tiling harmonic function $v(x, y)$ on the same square tiling such that $u(x, y) + v(x, y)$ is a nonzero graph harmonic function.

Existence of Pseudo-Conjugate

Conjecture

Given a tiling harmonic function $u(x, y)$ on some regular square tiling, there exists another tiling harmonic function $v(x, y)$ on the same square tiling such that $u(x, y) + v(x, y)$ is a nonzero graph harmonic function.

- The existence of such a "pseudo-conjugate" would be helpful for transferring properties of graph harmonic function to tiling harmonic functions.

Existence of Pseudo-Conjugate

Conjecture

Given a tiling harmonic function $u(x, y)$ on some regular square tiling, there exists another tiling harmonic function $v(x, y)$ on the same square tiling such that $u(x, y) + v(x, y)$ is a nonzero graph harmonic function.

- The existence of such a "pseudo-conjugate" would be helpful for transferring properties of graph harmonic function to tiling harmonic functions.
- We may be able to use the existence of this pseudo conjugate to prove a bounded difference between tiling and graph harmonic functions for all but a few special cases.

p-Tiling Harmonic Functions

- The difference in the definitions of tiling harmonic and graph harmonic functions lies in the energy functions that they minimize.

p-Tiling Harmonic Functions

- The difference in the definitions of tiling harmonic and graph harmonic functions lies in the energy functions that they minimize.
- In the classical case, the functions that satisfy $\mathrm{div}(|\nabla f|^{p-2}\nabla f) = 0$ where $p \geq 2$ form a class of functions called p-harmonic functions that have different properties from classical harmonic functions.

p-Tiling Harmonic Functions

- The difference in the definitions of tiling harmonic and graph harmonic functions lies in the energy functions that they minimize.
- In the classical case, the functions that satisfy $\text{div}(|\nabla f|^{p-2}\nabla f) = 0$ where $p \geq 2$ form a class of functions called p-harmonic functions that have different properties from classical harmonic functions.
- This is not the case for tiling harmonic functions and this highlights the difference between the Dirichlet energy function and the tiling harmonic energy function.

Equivalence of All p-Tiling Harmonic Functions

Conjecture

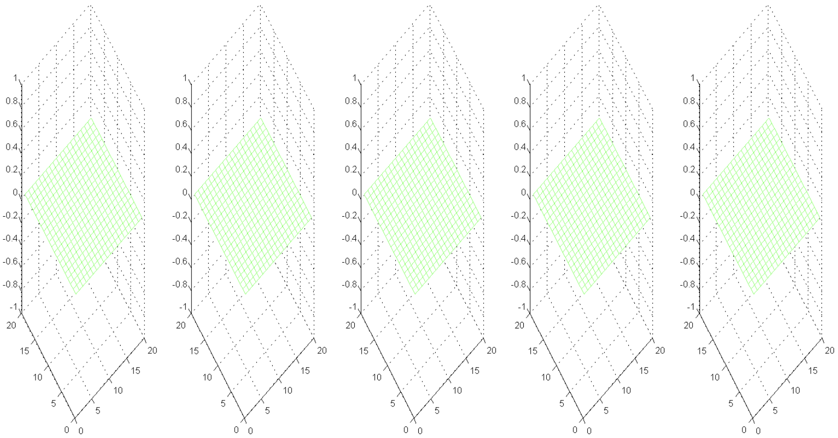
Given a fixed set of boundary values on a regular square tiling T, the function f that minimizes the p-energy

$$E_p(f, T) = \sum_{S \in T} \operatorname{osc}^p(S)$$

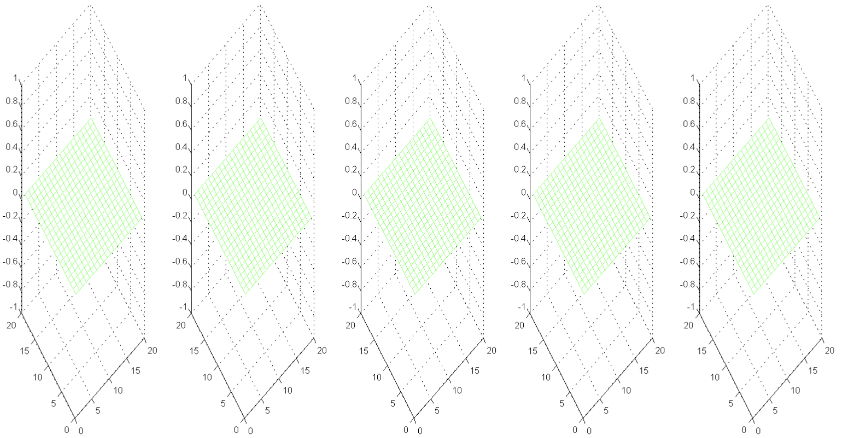
is fixed where p is an integer that is at least 2.

- This conjecture is strongly supported by experimental evidence for $p > 2$, but it appears that for non integral p, the p-tiling harmonic functions are not equivalent.

Experimental Evidence



Experimental Evidence



- The difference between the p-tiling harmonic function and the standard tiling harmonic function is 0 for $p = 3, 4, 5, 6, 7$ shown above.

Goals for Future Research

- We still want to find a specific definition of a Tiling analytic function that does not rely on the tiling harmonic computer algorithm to fill in the interior values of the conjugate.
- We wish to prove the aforementioned conjectures and use them to transfer properties from the graph harmonic case to the tiling harmonic case.

Conjecture

Any nonnegative tiling harmonic function in the upper half plane that vanishes on the x-axis must be of the form cy for some real number c.

Conjecture

A bounded tiling harmonic function on the regular lattice grid must be constant.

Acknowledgements

I would like to thank:

- The MIT PRIMES Program
- Prof. Sergiy Merenkov, CCNY-CUNY
- Dr. Tanya Khovanova
- and My Parents